Maths for Computing
Assignment 1

1. *(5 marks)* Construct a truth table for the following compound propositions.

   a) $p \to (\neg q \vee r)$

   Solution:

p	q	r	$\neg q$	$\neg q \vee r$	$p \to (\neg q \vee r)$
F	F	F	T	T	T
F	F	T	T	T	T
F	T	F	F	F	T
F	T	T	F	T	T
T	F	F	T	T	T
T	F	T	T	T	T
T	T	F	F	F	F
T	T	T	F	T	T

   b) $(p \to q) \vee (\neg p \to r)$

   Solution:

p	q	r	$\neg p$	$p \to q$	$\neg p \to r$	$(p \to q) \vee (\neg p \to r)$
F	F	F	T	T	F	T
F	F	T	T	T	T	T
F	T	F	T	T	F	T
F	T	T	T	T	T	T
T	F	F	F	F	T	T
T	F	T	F	F	T	T
T	T	F	F	T	T	T
T	T	T	F	T	T	T

2. *(5 marks)* Write each of these statements in the form "if p, then q" in English.

   a) I will remember to send you the address only if you send me an e-mail message.

   Solution:

If I remember to send you the address, then you send me an e-mail message.

b) To be a citizen of this country, it is sufficient that you were born in the United States.

Solution:

If you were born in the US, then you are a citizen of the country.

c) If you keep your textbook, it will be a useful reference in your future courses.

Solution:

If you keep your textbook, then it will be a useful reference in your future courses.

d) The RedWings will win the Stanley Cup if their goalie plays well.

Solution:

If RedWings' goalie plays well, then the RedWings will win the Stanley Cup.

e) That you get the job implies that you had the best credentials.

Solution:

If you get the job, then you had the best credentials.

Note: Solutions are not direct replacement of p and q from the given statements in "if p, then q" form. This is done to make sure that the grammar is correct. The intention behind this exercise was to check whether you understand $p \rightarrow q$ in different forms. Therefore, if you have correctly spotted p and q in the sentences, you will get full marks even if your sentence is not grammatically correct.

3. *(5 marks)* Show that the following pairs are logically equivalent.

a) $(p \rightarrow q) \wedge (p \rightarrow r)$ and $p \rightarrow (q \wedge r)$

Solution:

Both compound propositions are logically equivalent because their truth values are the same for all possible truth values of atomic propositions as shown in the below truth table.

p	q	r	$p \rightarrow q$	$p \rightarrow r$	$q \wedge r$	$(p \rightarrow q) \wedge (p \rightarrow r)$	$p \rightarrow (q \wedge r)$
F	F	F	T	T	F	T	T
F	F	T	T	T	F	T	T
F	T	F	T	T	F	T	T
F	T	T	T	T	T	T	T
T	F	F	F	F	F	F	F
T	F	T	F	T	F	F	F
T	T	F	T	F	F	F	F
T	T	T	T	T	T	T	T

b) $\neg p \rightarrow (q \rightarrow r)$ and $q \rightarrow (p \vee r)$

Solution:

Both compound propositions are logically equivalent because their truth values are the same for all possible truth values of atomic propositions as shown in the below truth table.

p	q	r	$p \vee r$	$q \rightarrow r$	$\neg p$	$\neg p \rightarrow (q \rightarrow r)$	$q \rightarrow (p \vee r)$
F	F	F	F	T	T	T	T
F	F	T	T	T	T	T	T
F	T	F	F	F	T	F	F
F	T	T	T	T	T	T	T
T	F	F	T	T	F	T	T
T	F	T	T	T	F	T	T
T	T	F	T	F	F	T	T
T	T	T	T	T	F	T	T

4. *(5 marks)* A collection of logical operator is called functionally complete if every compound proposition is logically equivalent to a compound proposition involving only these logical operators. Show that $\neg$ and $\wedge$ form a functionally complete collection of logical operator.

Solution: It is sufficient to show that $p \vee q$, $p \rightarrow q$, and $p \leftrightarrow q$ can be expressed using p, q, $\neg$, and $\wedge$. In other words, we need to form compound propositions using p, q, $\neg$, and $\wedge$ that are equivalent to $p \vee q$, $p \rightarrow q$, and $p \leftrightarrow q$, respectively.

Here are they:

1. $p \vee q \equiv \neg(\neg p \wedge \neg q)$
2. $p \rightarrow q \equiv \neg(p \wedge \neg q)$
3. $p \leftrightarrow q \equiv \neg(p \wedge \neg q) \wedge \neg(q \wedge \neg p)$

You should also show the logical equivalence of all three pairs using truth table or without using it. I am leaving that here.

Note: There can be multiple answers, but, if you have given correct logical equivalent compound propositions and shown their logical equivalence, your answer is correct.

5. *(7.5 (= 2.5 + 5) marks)* Show that the following pairs are logically equivalent without using truth table.

a) $\neg(p \vee (\neg p \wedge q)) \equiv \neg p \wedge \neg q$

Solution:

$\neg(p \vee (\neg p \wedge q))$

$\equiv \neg((p \vee \neg p) \wedge (p \vee q))$ (Distributive law)

$\equiv \neg(T \wedge (p \vee q))$ (Negation law)

$\equiv \neg((p \vee q) \wedge T)$ (Commutative law) *(I hope you did not forget this step. :))*

$\equiv \neg(p \vee q)$ (Identity law)

$\equiv \neg p \wedge \neg q$ (De Morgan's law)

b) $((p \rightarrow q) \wedge (q \rightarrow r)) \rightarrow (p \rightarrow r) \equiv T$

Solution: (This turned out to be a lot more difficult than I expected. Kudos to those who correctly solved it. Since this is extremely lengthy, I will be skipping a few obvious steps or apply more than one rule of inference in one step.)

$((p \rightarrow q) \wedge (q \rightarrow r)) \rightarrow (p \rightarrow r)$

$\equiv ((\neg p \vee q) \wedge (\neg q \vee r)) \rightarrow (\neg p \vee r)$ (Using $p \rightarrow q \equiv \neg p \vee q$)

$\equiv \neg((\neg p \vee q) \wedge (\neg q \vee r)) \vee (\neg p \vee r)$ (Using $p \rightarrow q \equiv \neg p \vee q$)

$\equiv ((p \wedge \neg q) \vee (q \wedge \neg r)) \vee (\neg p \vee r)$ (De Morgan's law)

$\equiv (((p \wedge \neg q) \vee (q \wedge \neg r)) \vee \neg p) \vee r$ (Associative law)

$\equiv (\neg p \vee ((p \wedge \neg q) \vee (q \wedge \neg r))) \vee r$ (Commutative law)

$\equiv ((\neg p \vee (p \wedge \neg q)) \vee (q \wedge \neg r)) \vee r$ (Associative law)

$\equiv (((\neg p \vee p) \wedge (\neg p \vee \neg q)) \vee (q \wedge \neg r)) \vee r$ (Distributive law)

$\equiv ((T \wedge (\neg p \vee \neg q)) \vee (q \wedge \neg r)) \vee r$ (Negation law)

$\equiv ((\neg p \vee \neg q) \vee (q \wedge \neg r)) \vee r$ (Identity law)

$\equiv (\neg p \vee (\neg q \vee (q \wedge \neg r))) \vee r$ (Associative law)

$\equiv (\neg p \vee ((\neg q \vee q) \wedge (\neg q \vee \neg r))) \vee r$ (Distributive law)

$\equiv (\neg p \vee (T \wedge (\neg q \vee \neg r))) \vee r$ (Negation law)

$\equiv (\neg p \vee (\neg q \vee \neg r)) \vee r$ (Identity law)

$\equiv ((\neg p \vee \neg q) \vee \neg r) \vee r$ (Associative law)

$\equiv (\neg p \vee \neg q) \vee (\neg r \vee r)$ (Identity law)

$\equiv (\neg p \vee \neg q) \vee T$ (Negation law)

$\equiv T$ (Domination law)

6. *(5 marks)* Show that the following pairs are not logically equivalent

a) $\exists x P(x) \rightarrow \exists x Q(x)$ and $\exists x Q(x) \rightarrow \exists x P(x)$

Solution:

Let $P(x) = x$ is a prime number, $Q(x) = x$ is a composite number, and domain be the set of prime numbers. Since $\exists x P(x)$ is true and $\exists x Q(x)$ is false, $\exists x P(x) \rightarrow \exists x Q(x)$ is false but $\exists x Q(x) \rightarrow \exists x P(x)$ is true.

b) $\forall x P(x) \vee \forall x Q(x)$ and $\forall x (P(x) \vee Q(x))$

Solution:

Let $P(x) = x$ is a prime number, $Q(x) = x$ is a composite number, and domain be the set of positive integers. Since every positive integer is either a prime or a composite, $\forall x (P(x) \vee Q(x))$ is true. But, $\forall x P(x)$ is false because not all positive integers are prime and $\forall x Q(x)$ is false because not all positive integers are composite. Hence, $\forall x P(x) \vee \forall x Q(x)$ is false as well.

Note that to show that they are not logically equivalent you need to choose predicates and domain for which both WFFs of a pair have opposite truth value. You should choose exactly one domain for a variable that is present in both WFFs.

7. *(3 marks)* Find the truth value of the following propositions where the domain is the set of positive integers. Justify your answer briefly.

a) $\forall x \exists y (x = 1/y)$

Solution:

This is false. For $x = 2$, there is no y such that $x = 1/y$, because for every y in the domain $1/y \leq 1$.

b) $\forall x \exists y (y^2 - x < 100)$

Solution:

This is true. First rearrange the inequality and form an equivalent proposition $\forall x \exists y (y^2 - 100 < x)$. This proposition is true because for every positive integer x, there exists a y, say $y = 1$, for which the inequality is true as the LHS is a negative number.

c) $\forall x \forall y (x^2 \neq y^3)$

Solution:

This is false. Take $x = 8$ and $y = 4$. Or simply $x = y = 1$.

8. *(10.5 (= 3.5 * 3) marks)* Show that the following arguments are valid. (Write all the steps with reasons.)

a) Premises: $p \to (\neg r \to \neg q)$, $\neg r$. Conclusion: $\neg(p \land q)$

Solution:

1. $p \to (\neg r \to \neg q)$ (Premise)
2. $\neg p \lor (\neg r \to \neg q)$ (Using $p \to q \equiv \neg p \lor q$ on 1)
3. $\neg p \lor (r \lor \neg q)$ (Using $p \to q \equiv \neg p \lor q$ on 2)
4. $(\neg p \lor r) \lor \neg q$ (Associative law on 3)
5. $(r \lor \neg p) \lor \neg q$ (Commutative law on 4)
6. $r \lor (\neg p \lor \neg q)$ (Associative law on 5)
7. $\neg r$ (Premise)
8. $\neg p \lor \neg q$ (Disjunctive syllogism on 6 and 7)
9. $\neg(p \land q)$ (De Morgan's law on 8)

b) Premises: $(p \lor (q \lor r)) \land (p \leftrightarrow s)$, $q \to t$, $t \to \neg q$. Conclusion: $p \lor r$

Solution:

1. $(p \lor (q \lor r)) \land (p \leftrightarrow s)$ (Premise)
2. $p \lor (q \lor r)$ (Simplification on 1)
3. $(p \lor q) \lor r$ (Associative law on 2)
4. $(q \lor p) \lor r$ (Commutative law on 3)
5. $q \lor (p \lor r)$ (Associative law on 4)
6. $q \to t$ (Premise)
7. $\neg q \lor t$ (Using $p \to q \equiv \neg p \lor q$)
8. $t \to \neg q$ (Premise)
9. $\neg t \lor \neg q$ (Using $p \to q \equiv \neg p \lor q$)
10. $t \lor \neg q$ (Commutative law on 7)
11. $\neg q \lor \neg q$ (Resolution on 9 and 10)
12. $\neg q$ (Idempotent law on 11)
13. $p \lor r$ (Disjunctive syllogism on 5 and 12)

c) Premises: $\forall x(P(x) \lor Q(x))$, $\forall x((\neg P(x) \land Q(x)) \to R(x))$.
Conclusion: $\forall x(\neg R(x) \to P(x))$ (Domain for all quantifiers are the same.)

Solution:

1. $\forall x(P(x) \lor Q(x))$ (Premise)
2. $P(c) \lor Q(c)$ for an arbitrary c in the domain (Universal instantiation of 1)

3. $\forall x((\neg P(x) \wedge Q(x)) \rightarrow R(x))$ (Premise)

4. $(\neg P(c) \wedge Q(c)) \rightarrow R(c)$ for an arbitrary c in the domain (Universal instant. of 3)

5. $\neg(\neg P(c) \wedge Q(c)) \vee R(c)$ for an arbitrary c in … (Using $p \rightarrow q \equiv \neg p \vee q$ on 4)

6. $(P(c) \vee \neg Q(c)) \vee R(c)$ for an arbitrary c in the domain (De Morgan's law on 5)

7. $(\neg Q(c) \vee P(c)) \vee R(c)$ for an arbitrary c in the domain (Commutative law on 6)

8. $\neg Q(c) \vee (P(c) \vee R(c))$ for an arbitrary c in the domain (Associative law on 7)

9. $Q(c) \vee P(c)$ for an arbitrary c in the domain (Commutative law on 2)

10. $P(c) \vee (P(c) \vee R(c))$ for an arbitrary c in the domain (Resolution on 8 and 9)

11. $(P(c) \vee P(c)) \vee R(c)$ for an arbitrary c in the domain (Associative law on 10)

12. $P(c) \vee R(c)$ for an arbitrary c in the domain (Idempotent law on 11)

13. $R(c) \vee P(c)$ for an arbitrary c in the domain (Commutative law on 12)

14. $\neg R(c) \rightarrow P(c)$ for an arbitrary c in the domain (Using $p \rightarrow q \equiv \neg p \vee q$ on 13)

15. $\forall x(\neg R(x) \rightarrow P(x))$ (Universal generalization of 14)